

BKM Ch. 6

Outline

I. Risk, Speculation and Gambling

- ◇ **Speculation** is the assumption of considerable investment risk to obtain commensurate gain
- ◇ **Gambling** is to bet or wager on an uncertain outcome

The **key difference** between speculation and gambling is that gambling lacks a “commensurate gain.” For example, if a risky investment has a risk premium equal to zero (known as a **fair game**), then it is a gamble because there is no expected gain to compensate for the risk. To turn a gamble into a speculative venture, there must be an adequate risk premium present.

It's possible for a gamble to appear speculative. For **example**, suppose we have the following situation:

- ◇ Investor A will pay Investor B \$100 if the value of $\mathcal{L}1$ is greater than \$1.40 one year from now
- ◇ Investor B will pay Investor A \$100 if the value of $\mathcal{L}1$ is less than \$1.40 one year from now
- ◇ Thus, there are two possible outcomes – the value of $\mathcal{L}1$ is greater than \$1.40 one year from now OR the value of $\mathcal{L}1$ is less than \$1.40 one year from now
- ◇ If each investor believes that the two outcomes have equal probabilities of occurring, the expected profit to both investors is zero. This means each investor is gambling
- ◇ If each investor assigns different probabilities for each outcome, the expected profit to both investors is non-zero. This means each investor is speculating

II. Risk Aversion and Utility Values

Risk-averse investors consider only risk-free or speculative prospects with positive risk premiums. Risk-averse investors also penalize the expected rate of return of a risky portfolio. As the risk increases, the penalty increases.

How do investors choose between portfolios of varying risk? Furthermore, how do investors quantify the rate at which they are willing to trade off return against risk? To answer these questions, we will assume the following:

- ◇ Each investor assigns a **utility** score to various portfolios based on the expected return and risk of those portfolios
- ◇ Higher utility scores are given to portfolios with more attractive risk-return profiles
- ◇ The utility score is defined as follows:

$$U = E(r) - \frac{1}{2}A\sigma^2$$

where U is the utility score, $E(r)$ is the expected return for a portfolio, A is the investor's risk aversion and σ^2 is the variance of returns for the portfolio

Notice that utility increases as the expected return increases and decreases as the variance (i.e. risk) increases. Also, notice that A essentially controls the impact of the variance on utility. More risk-averse investors have larger values of A , which represents the higher penalty they place on risky investments. When choosing portfolios, investors attempt to maximize utility.

Example

Three investors with risk aversion indices of $A_1 = 2$, $A_2 = 3.5$ and $A_3 = 5$ are choosing between the following three portfolios:

Portfolio	Expected Return	Risk (SD)
L (low risk)	7%	5%
M (medium risk)	9	10
H (high risk)	13	20

We are also told that the risk-free rate is 5%.

The utility scores for each portfolio by investors is as follows:

Investor Risk Aversion	Utility Score of Portfolio L	Utility Score of Portfolio M	Utility Score of Portfolio H
2.0	$0.0675 = 0.07 - \frac{1}{2}(2)(0.05^2)$	0.08	0.09
3.5	0.0656	0.0725	0.06
5.0	0.0638	0.0650	0.03

What do we notice in the table above?

- ◇ First, let's note that a risk-free investment (i.e. $\sigma^2 = 0$) would produce a utility score of 0.05 for each investor. Thus, any investment in a risky portfolio must produce a utility score above 0.05 to be chosen
- ◇ The high-risk investor (i.e. large A) chooses Portfolio H. This is not surprising given that it doesn't penalize the large variance of Portfolio H as much as the other two investors

- ◇ The low- and medium-risk investors choose Portfolio M. Even the most risk-averse investor chooses Portfolio M over Portfolio L

The utility score is also known as the **certainty equivalent rate** of return. The certainty equivalent rate is the rate that a risk-free investment would need to offer to provide the same utility score as the risky portfolio. A portfolio is **only desirable if its certainty equivalent rate** is greater than the risk-free rate. This is the same thing we said in the first bullet above.

Risk-neutral investors with $A = 0$ ignore risk and choose portfolios based entirely on the expected return. In other words, risk-neutral investors do not penalize risk.

Risk lovers with $A < 0$ adjust the expected return upward as risk increases. In other words, they do not penalize risk. They actually consider it an enjoyable benefit.

We can depict the relationship between risk and return by plotting portfolios on a graph with axes measuring the expected return (y -axis) and the standard deviation of return (x -axis). Portfolios with equal utility live on a special curve within the graph known as an **indifference curve**. On this curve, all portfolios are equally preferred for a particular investor.

Using our example above, let's plot the indifference curves for Portfolio L for two investors: $A_1 = 2.0$ and $A_2 = 3.5$. To determine the (x, y) points for the indifference curves, we set up a table of standard deviations and solve for the expected returns that produce a utility equal to Portfolio L's utility for each investor. For the first investor ($A_1 = 2.0$), the utility (or certainty equivalent rate) was 0.0675. For the second investor ($A_2 = 3.5$), the utility was 0.0656. Then, the (x, y) points are as follows:

σ_P	Expected Return ($E[r_P]$)	
	Investor 1	Investor 2
0.00	6.75%	6.56%
0.02	6.79%	6.63%
0.04	6.91%	6.84%
0.05	7.00%	7.00%
0.06	7.11%	7.19%
0.08	7.39%	7.68%

For a portfolio P , let's look at the calculation for the expected return $E[r_P]$ when the standard deviation of the returns $\sigma_P = 0.02$:

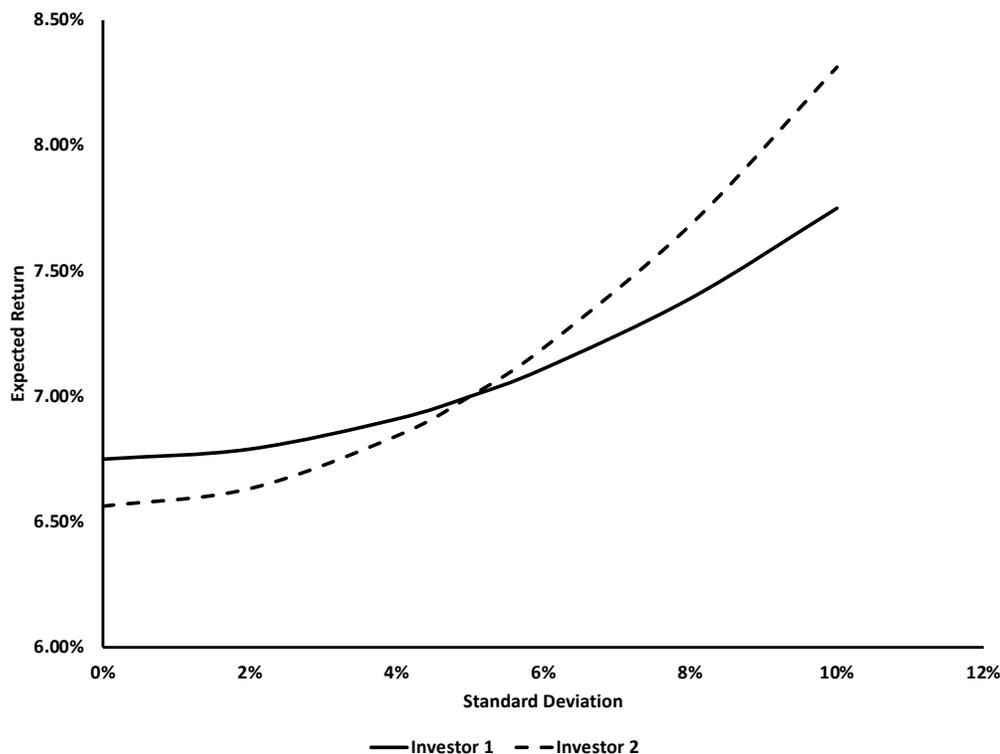
- ◇ Investor 1

- We want Utility = $0.0675 = E[r_P] - \frac{1}{2}(2)(0.02^2)$. Then, $E[r_P] = 0.0679$. This means that for Investor 1, a portfolio with $E[r_P] = 0.0679$ and $\sigma_P = 0.02$ is equally preferred to Portfolio L where $E[r_L] = 0.07$ and $\sigma_L = 0.05$

◇ Investor 2

- We want $Utility = 0.0656 = E[r_P] - \frac{1}{2}(3.5)(0.02^2)$. Then, $E[r_P] = 0.0663$. This means that for Investor 2, a portfolio with $E[r_P] = 0.0663$ and $\sigma_P = 0.02$ is equally preferred to Portfolio L where $E[r_L] = 0.07$ and $\sigma_L = 0.05$

The indifference curves are as follows:



What conclusions can we draw from the graph above?

- ◇ For the same portfolio, the y -intercept is lower for the investor with more risk-aversion (i.e. larger A)
- ◇ For the same portfolio, the y -intercepts are the utilities or certainty equivalent rates for each investor
- ◇ For the same portfolio, the investor with more risk-aversion will show a steeper curve. In other words, as risk increases, a more risk-averse investor requires relatively higher levels of return
- ◇ For the same portfolio, the indifference curves will intersect at the expected return and standard deviation of the portfolio. In this case, they intersect at (5%, 7%), which is the expected return and standard deviation for Portfolio L

In the example above, we are given A values to represent each investor's risk aversion. How might we go about estimating an investor's risk aversion?

- ◇ Design a questionnaire that allows investors to pinpoint specific levels of risk aversion coefficients
- ◇ Infer risk aversion coefficients based on empirical observations of risk and return for active investors
- ◇ Track behavior of individuals (ex. the purchase of insurance policies) to obtain average degrees of risk aversion

III. Capital Allocation Across Risky and Risk-Free Portfolios

For an investment portfolio, how much of the portfolio should be placed in risk-free securities vs. risky securities? Let P be the investor's portfolio of risky assets and F be the investor's portfolio of risk-free assets.

Let's look at an **example**. Given the following:

- ◇ The total market value of a portfolio is \$300,000
- ◇ \$90,000 of the portfolio is invested in risk-free assets
- ◇ \$210,000 of the portfolio is invested in risky assets, where \$113,400 is invested in equities (E) and \$96,600 is invested in bonds (B)

The weights (w) within the risky portfolio are as follows:

- ◇ $w_E = \frac{113,400}{210,000} = 54\%$
- ◇ $w_B = \frac{96,600}{210,000} = 46\%$

The weight of the risky portfolio, P , in total is $y = \frac{210,000}{300,000} = 70\%$. Thus, the weight of the risk-free portfolio, F , is 30%.

Suppose the owner of the portfolio would like to **decrease risk by reducing the allocation to the risky portfolio** from $y = 0.70$ to $y = 0.56$. In this case, the risky portfolio would decrease from \$210,000 to $\$300,000(0.56) = \$168,000$. This means that $\$210,000 - \$168,000 = \$42,000$ of the risky portfolio would be sold, with the proceeds used to purchase more risk-free securities. It's important to note that the weights of the risky portfolio remain unchanged after the sale. Thus, the owner would sell $\$42,000(0.54) = \$22,680$ of E and $\$42,000(0.46) = \$19,320$ of B . The risk-free portfolio would increase to $\$90,000 + \$42,000 = \$132,000$. We can confirm that the weights of the risk portfolio remain unchanged:

- ◇ $w_E = \frac{113,400 - 22,680}{210,000 - 42,000} = 54\%$
- ◇ $w_B = \frac{96,600 - 19,320}{210,000 - 42,000} = 46\%$

Although we have not changed the composition of the risky portfolio, we have reduced the risk of the complete portfolio by shifting to more risk-free securities.

We mentioned the composition of the risky portfolio. What about the risk-free portfolio? When we think of risk-free assets, the most common things to come to mind are **Treasury Bills**. Due to their short-term nature, they are not sensitive to interest rate fluctuations or inflation uncertainty. In addition, they are issued by the government which means they should be default-free. In practice, other Treasury and U.S. agency securities, as well as repurchase agreements, are considered risk-free. Again, this is driven by their short-term nature as well as lack of default and credit risk.

IV. Portfolios of One Risky Asset and a Risk-Free Asset

In the prior section, we discussed an example where the owner of a portfolio decreased the allocation to the risky portfolio from $y = 0.70$ to $y = 0.56$. In the next two sections, we will examine how we determine the optimal allocation to the risky portfolio.

The **expected rate of return on the complete portfolio** is as follows:

$$\begin{aligned} E(r_C) &= yE(r_P) + (1 - y)r_f \\ &= r_f + y[E(r_P) - r_f] \\ &= r_f + y \times \text{Risk Premium} \end{aligned}$$

where $E(r_C)$ is the expected return for the complete portfolio C , $E(r_P)$ is the expected return for the risky portfolio P , y is the allocation of the complete portfolio to the risky portfolio and r_f is the risk-free asset. In the equation above, the base rate of return for any portfolio is the risk-free rate. Beyond the risk-free rate, the portfolio is expected to earn a proportion y of the risky portfolio's return over the risk-free rate.

The **standard deviation on the complete portfolio** is as follows:

$$\sigma_C = y\sigma_P$$

where σ_C is the standard deviation of the complete portfolio and σ_P is the standard deviation of the risky portfolio.

We can rewrite the expected return of the complete portfolio as a function of the standard deviation of the complete portfolio using the fact that $y = \frac{\sigma_C}{\sigma_P}$:

$$\begin{aligned} E(r_C) &= r_f + y[E(r_P) - r_f] \\ &= r_f + \frac{\sigma_C}{\sigma_P}[E(r_P) - r_f] \\ &= r_f + \frac{[E(r_P) - r_f]}{\sigma_P}\sigma_C \\ &= r_f + S\sigma_C \end{aligned}$$

where $S = \frac{[E(r_P) - r_f]}{\sigma_P}$ is the **Sharpe ratio** (also known as the reward-to-volatility ratio). In the equation above, we have shown that the expected return of the complete portfolio as a function of

its standard deviation is a straight line with a y -intercept equal to the risk-free rate and a slope equal to the Sharpe Ratio. This straight line is known as the **capital allocation line (CAL)** and represents **the investment opportunity set**, which is the set of all risk-return combinations available to investors.

The Sharpe Ratio can be interpreted as the average percentage excess return over the risk-free rate for every 1% of additional standard deviation. For example, if the Sharpe Ratio is 0.40, then stock investors obtain a 0.40% average excess return over the risk-free rate for every 1% of additional standard deviation.

Example

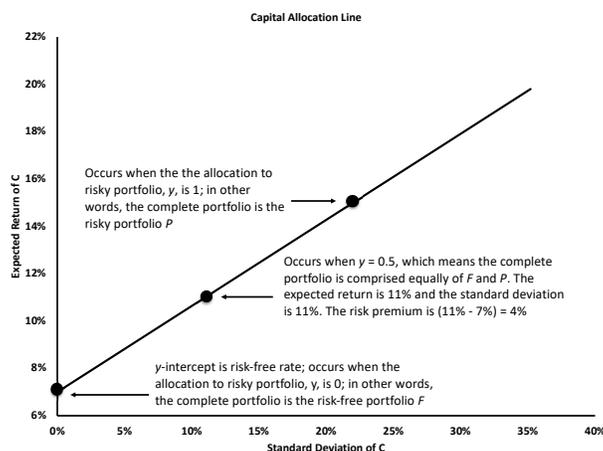
Given the following:

$$\diamond E(r_P) = 15\%$$

$$\diamond \sigma_P = 22\%$$

$$\diamond r_f = 7\%$$

Then, the Sharpe Ratio $S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.15 - 0.07}{0.22} = \frac{8}{22}$. Thus, the CAL is $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}\sigma_C$. Let's go ahead and graph this line:



We notice the following in the graph above:

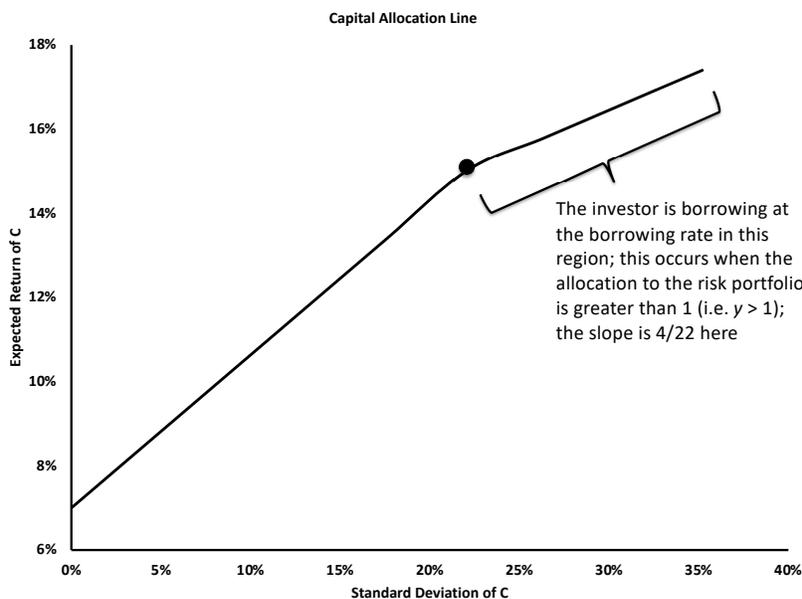
- \diamond Recall that $E(r_C)$ can be written as $E(r_C) = yE(r_P) + (1 - y)r_f$. When $y = 0$, the complete portfolio consists solely of the risk-free portfolio F . This occurs at the y -intercept
- \diamond When $y = 1$, the complete portfolio consists solely of the risky portfolio P . At this point on the graph, $\sigma_C = y\sigma_P = (1)(0.22) = 22\%$ and $E(r_C) = (1)(0.15) + (1 - 1)(0.07) = 15\%$. Alternatively, we could have calculated $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}(0.22) = 15\%$
- \diamond When $y = 0.5$, the complete portfolio consists equally of F and P . At this point on the graph, $\sigma_C = y\sigma_P = 0.5(0.22) = 11\%$ and $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}(0.11) = 11\%$

◇ The line extends past the point where the complete portfolio consists solely of P . How is that possible? Can $y > 1$? Yes! If investors can borrow at the risk-free rate of $r_f = 7\%$, then they can construct portfolios to the right of P on the CAL. As an example:

- The market value of a risky portfolio is \$300,000
- An investor borrows an additional \$120,000
- This is a **levered position** in the risky asset since some of the financing is coming from borrowed money. In this case, $y = \frac{420,000}{300,000} = 1.4$
- Thus, the amount allocated to the risk-free asset is $1 - y = 1 - 1.4 = -0.4$. This reflects the fact that investor has a borrowing position in the risk-free asset
- Since levered positions are further to the right on the CAL, they have higher standard deviations than un-levered positions. This is not surprising given the additional uncertainty

Regarding the borrowing position above, points to the right of P on the CAL assume that the investor can borrow at the risk-free rate. In reality, non-government investors cannot borrow at the risk-free rate since they have some level of default risk. Thus, a non-government borrower's rate will always be higher than the risk-free rate. This creates a "kink" on the CAL at the point where $y = 1$. Beyond this point, the slope of the CAL changes to reflect the borrower's rate.

Suppose the borrower's rate is $r_f^B = 11\%$. Then, beyond the point where $y = 1$, the Sharpe Ratio $S_{y>1} = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.15 - 0.11}{0.22} = \frac{4}{22}$. This creates the following graph:



To calculate the "kinked" portion of the CAL, we use $\sigma_C = y\sigma_P$ and $E(r_C) = r_f^B + S_{y>1}\sigma_C$. Notice that these are the same formulas as above. However, we have replaced r_f with r_f^B and

S with $S_{y>1}$. For **example**, suppose $y = 1.2$. Then, $\sigma_C = y\sigma_P = 1.2(0.22) = 26.4\%$ and $E(r_C) = r_f^B + S_{y>1}\sigma_C = 0.11 + \frac{4}{22}(0.264) = 15.8\%$.

V. Risk Tolerance and Asset Allocation

This section will discuss how an investor determines where on the CAL to reside. For any investor, the goal is to choose the portfolio that maximizes utility. Recall that utility $U = E(r) - \frac{1}{2}A\sigma^2$. Let's consider an example:

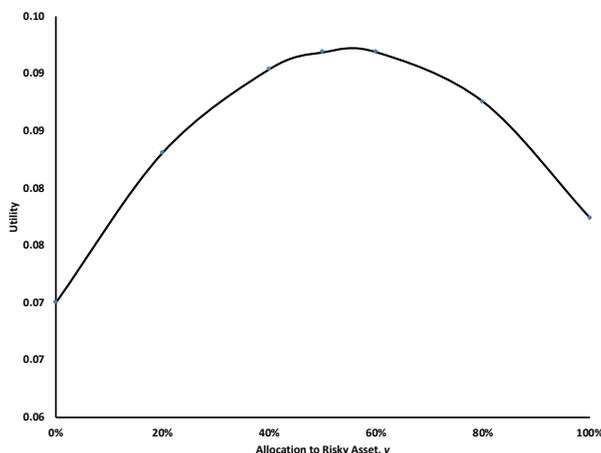
$$\diamond E(r_P) = 15\%$$

$$\diamond \sigma_P = 22\%$$

$$\diamond r_f = 7\%$$

$$\diamond A = 3$$

Notice that this is the same setup as our example in the prior section. Let's plot the utility U as a function of the allocation y to the risky portfolio P :



The graph above shows that utility increases with y up until a certain point. Then, the volatility outweighs the expected return causing utility to decrease.

We want to choose the y that corresponds to the maximum point on the utility curve. As you might have imagined, this is a calculus problem. We need to take the derivative of $U = E(r) - \frac{1}{2}A\sigma^2$ with respect to y and set it equal to 0. Using our formulas from the prior section, we can rewrite U as $U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$. The derivative of U with respect to y is as follows:

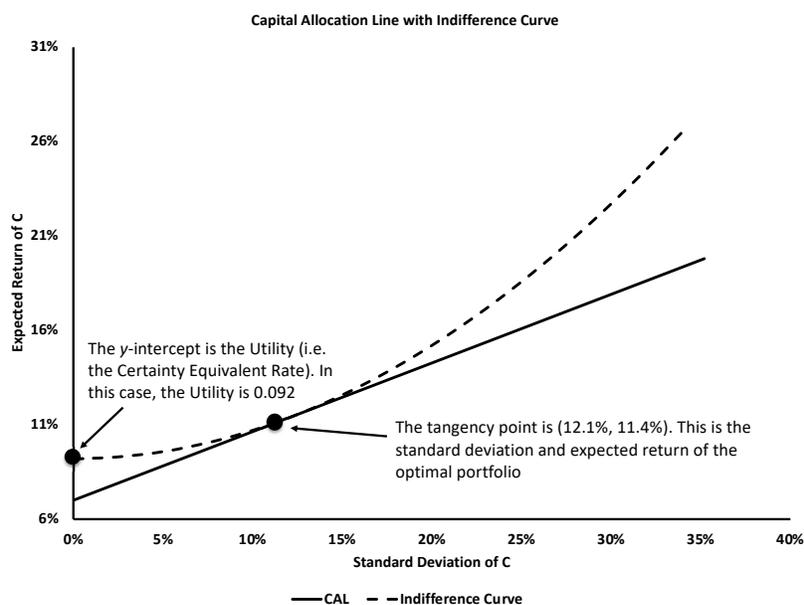
$$\frac{dU}{dy} = E(r_P) - r_f - Ay\sigma_P^2$$

If we set $E(r_P) - r_f - Ay\sigma_P^2 = 0$, we find that the optimal position for risk-averse investors in the risky asset is as follows:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

Using our **example**, we find that $y^* = \frac{0.15-0.07}{3(0.22^2)} = 0.55$. Thus, to maximize utility, an investor with $A = 3$ should allocate 55% to the risky portfolio P . The utility at this allocation level is $U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2 = 0.07 + 0.55(0.15 - 0.07) - \frac{1}{2}(3)(0.55^2)(0.22^2) = 0.092$. The expected return is $E(r_C) = yE(r_P) + (1 - y)r_f = 0.55(0.15) + (1 - 0.55)(0.07) = 11.4\%$, and the standard deviation is $\sigma_C = y\sigma_P = 0.55(0.22) = 12.1\%$.

Another way to find y^* is using indifference curves. Recall from earlier that indifference curves plot all risk-return combinations that produce the same utility for a particular investor. For the same investor, higher indifference curves (i.e. higher y -intercepts) are better because they represent higher utilities. The **key result** is that the **highest indifference curve that still touches the CAL** (i.e. tangent to the CAL) **identifies the optimal complete portfolio**. In particular, the tangency point **corresponds to the standard deviation and expected return of the optimal complete portfolio**. Here is the capital allocation line from our example, along with the highest indifference curve that touches the CAL:



In the graph above, the y -intercept of the indifference curve is 0.092. The tangency point is (12.1%, 11.4%). These are the same figures we arrived at above for U , σ_C and $E(r_C)$.

The last thing to note from the graph above is that the choice for y^* is determined by **two things**:

- (1) Risk Aversion – This is captured by the slope of the indifference curve. Higher risk aversion (i.e. larger A) results in a steeper indifference curve
- (2) Sharpe Ratio – This is captured by the slope of the opportunity set or CAL. A different Sharpe Ratio impacts the tangency point of the highest indifference curve

There is one more thing to note from this section. In all of the preceding discussion, we have treated standard deviation as the appropriate measure of risk. This **implicitly assumes that returns are normally distributed**. However, this is not necessarily the case. In cases of non-normality with fat tails, investors should consider measuring the Value at Risk (VaR) and reducing their allocation to the risky portfolio.

VI. Passive Strategies: The Capital Market Line

A **passive strategy** avoids any direct or indirect security analysis. It involves investment in two portfolios:

- (1) Risk-free short-term T-bills (or alternatively, a money market fund)
- (2) A well-diversified portfolio of common stocks that acts similarly to a broad market index. This is known as a “neutral” diversification strategy. Under this approach, if a stock comprises 3% of the market value of all listed stocks, then it should comprise 3% of the investor’s risky portfolio

The capital allocation line that results from holding a mix of the risk-free asset and a broad index of common stocks is known as the **capital market line**. This is the line produced by a passive strategy.

Why might one pursue a passive strategy over an active strategy?

- ◇ Passive strategies are less expensive than active strategies
 - If an investor pursues an active strategy on his or her own, there is time and cost to acquire the information needed to generate an optimal active portfolio of risky assets. If an investor hires a professional to pursue an active strategy, there is the fee owed to the professional
 - Passive strategies only require the small costs to purchase T-bills and modest fees to the management company
- ◇ Passive strategies enjoy the free-rider benefit
 - When there are a number of active investors, they quickly bid up prices of undervalued assets (through buying) and force down prices of overvalued assets (through selling). This leads to most assets being fairly priced at any given time
 - This means that a well-diversified portfolio of common stock should be a fair buy, resulting in a passive strategy that may not be any worse than an active strategy